Diagnosis under compound effects and multiple causes by means of the conditional causal possibility approach

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Abstract

The paper addresses uncertain reasoning on a causal model given by two layered networks, where nodes in one layer express possible causes and those in the other are possible effects. Uncertainty of causalities is expressed in a novel manner, i.e. by Conditional Causal Possibilities. The expression has two advantages over the conventional way with conditional possibilities: it expresses the exact degrees of possibility of causalities, and the number of necessary conditional causal possibilities is far smaller than that of conditional possibilities. However, it has a weakness that it cannot handle causalities with compound effects such as synergistic and canceling effects by multiple causes.

The paper discusses the weakness and proposes a solution. First, it discusses how to deal with the compound effects and proposes a new causal model with conditional causal possibilities by multiple causes. Then, it defines a causality consistency problem that calculates possibility of a hypothesis given some observed events, and shows a way to solve the problem.

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1. Introduction

Belief networks (BNs), which are a type of well-established probabilistic networks, are attracting a great deal of attention [15,16]. However, their application is limited in spite of the popularity of the theoretical research due to the fact that problems of learning BNs and probabilistic inference using them are NP-hard [2,3,14], while the recent research is focusing on reduction of the complexity [12,14]. BNs also have a problem to need a huge number of conditional probabilities as a priori knowledge: as many as combinations of states of all possible causes for each effect.

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A possible alternative of the ordinary BN is a probabilistic causal model with conditional causal probabilities proposed by [17,18], if the objective system can be modeled by the “noisy-or-gate” where causes generate effects disjunctively [16]. In fact, it seems that many diagnostic problems in industry can be modeled by the noisy-or-gate, considering that a more restrictive assumption such as single fault assumption has been common in the field. One of the advantages of the model is that it needs only as many conditional causal probabilities as the number of possible causes for each effect. Another is that the conditional causal probabilities might express the probabilities of causalities that the human recognizes in her mind [1,11,18].

Now, probability is a ratio scale of uncertainty. The quantity of its value has an exact meaning of its own, and should be evaluated accurately within its tolerance. In solving real world problems, however, it is frequently the case where not enough data can be gathered to determine the probabilities statistically and rough subjective probabilities are used instead.

There is another scale of uncertainty called possibility [6,24]. Since possibility is essentially an ordinal scale [7,8], reasoning based on the theory is insensitive to some errors of uncertainty. Thus, it might be suitable for problems with not enough data to calculate reliable probabilities. Possibilistic causal reasoning has recently been studied in this context [4,9,19–23], and some of them employ conditional causal possibilities, which are a possibilistic version of conditional causal probabilities [19–22].

The advantages of using conditional causal possibilities are the same as those of using conditional causal probabilities. However, they have a weakness, as compensation of the advantages, that they cannot handle causalities with compound effects by multiple causes such as synergistic and canceling effects on uncertainty of causality [20]. The compound effect appears when multiple causes generate an effect at a higher/lower possibility than the possibility calculated under the assumption that the causes generate the effect disjunctively. When the possibility is higher, it is synergistic, and when it is lower, it is canceling. In such cases, approaches with conditional possibilities by combinations of states of all possible causes are applicable, however they need a huge amount of possibilistic knowledge and computation as mentioned before.

Looking at the actual applications, there might be many problems where the number of combinations of causes producing the compound effects is far smaller than that of all combinations of possible causes. If the problem is the case, a new approach incorporating the notion of compound effects in the conditional causal possibilities provides an alternative method to solve it with far less possibilistic knowledge and computation.

The paper addresses the problem with the compound effects. It discusses how to deal with the compound effects on possibilities, and proposes a new causal model with conditional causal possibilities by multiple causes. Then, it defines and solves the possibilistic causality consistency problem based on the proposed model.

In the next section, a causal model with conditional causal possibilities is described briefly. It is expanded to be able to deal with the compound effects in Sections 3 and 4. In Section 5, the causality consistency problem with compound effects is defined and its solution is discussed. Finally, a simple numerical example is shown in Section 6.

2. Causal model with conditional causal possibilities

A causal model with conditional causal possibilities, which was proposed in [19,22], is described briefly in this section.
2.1. Causal model

Let \( U = \{u_i \mid i = 1, \ldots, I \} \) and \( V = \{v_j \mid j = 1, \ldots, J \} \) be two disjoint sets of primitive events. Each element \( u_i \) of \( U \) is called a cause and each element \( v_j \) of \( V \) is called an effect, respectively. Then, let \( c_{i,j} \) (\( i = 1, \ldots, I, j = 1, \ldots, J \)) denote a causation event that should be read as “\( u_i \) arises and \( u_i \) actually causes \( v_j \).” This is an event proposed by Peng and Reggia [17,18] and used in [22], though a different notation \( v_j : u_i \) was employed in the papers. \( C \) is the set of all causation events, namely \( C = \{c_{i,j} \mid i = 1, \ldots, I, j = 1, \ldots, J \} \). These events in \( U \), \( V \) and \( C \) have a state representing presence or absence of the event. The states of presence are denoted by classical propositions \( u_i^+, v_j^+ \) and \( c_{i,j}^+ \), respectively, and are referred to as positive. Those of absence are denoted by their negations \( u_i^-, v_j^- \) and \( c_{i,j}^- \), and are called negative. For example, suppose that \( u_1 \) is an event named “engine-overheated.” Then, the positive state is denoted by a proposition \( u_1^+ \) meaning “the engine is overheated.” The negative state is denoted by \( u_1^- \) that means “the engine is not overheated.”

The causal model is built on classical formulae with the use of these propositions, and assumes that the following formula holds:

\[
c_{i,j}^+ \rightarrow u_i^+ \land v_j^+ ,
\]

where \( \land \) and \( \rightarrow \) are conjunction and implication, respectively.

The following formulae are easily derived from the above.

\[
c_{i,j}^+ \leftrightarrow c_{i,j}^+ \land u_i^+ \leftrightarrow c_{i,j}^+ \land v_j^+ \leftrightarrow c_{i,j}^+ \land u_i^+ \land v_j^+ ,
\]

\[
c_{i,j}^- \leftrightarrow u_i^- ,
\]

\[
c_{i,j}^+ \land v_j^- \leftrightarrow v_j^- ,
\]

where \( \leftrightarrow \) denotes equivalence. See [17,18,22] for further discussions about the causation event.

The next formula is assumed to hold:

\[
v_j^+ \leftrightarrow \bigvee_{i=1,I} c_{i,j}^+ ,
\]

where \( \bigvee \) denotes disjunction. This is called mandatory causation assumption [17,18]. The causal model with the assumption is called asymmetrically-valued causal model (AVC model) in [21,22]. In the model, the following are easily derived from formulae (1) and (5):

\[
v_j^- \leftrightarrow \bigwedge_{i=1,I} c_{i,j}^- ,
\]

\[
v_j^+ \rightarrow \bigvee_{i=1,I} u_i^+ .
\]

2.2. Conditional causal possibility

Let \( \pi(s_i) (i = 1, \ldots, n) \) be a possibility distribution on a universal set \( S = \{s_1, \ldots, s_n\} \). The distribution must satisfy \( \bigvee_i \pi(s_i) = 1 \), where \( \bigvee (\bigwedge) \) denotes max (min) when used for possibilities, while
it denotes disjunction (conjunction) for propositions in the paper. Then, a possibility measure \( \Pi(A) \), \( A \subseteq S \) could be defined using the distribution.

\[
\Pi(A) = \bigvee_{s_i \in A} \pi(s_i),
\]

(8)

where \( \Pi(A) = 0 \) if \( A \) is the empty set, and \( \Pi(A) = 1 \) if \( A = S \). \( \Pi(A) \lor \Pi(A^c) = 1 \) for any \( A \) and its complement \( A^c \). In the paper, notation such as \( \Pi(\bigvee_{s_i \in A} s_i) \) is used instead of \( \Pi(A) \) in order to handle possibility of events denoted by logical expressions.

Let \( e_k \) (\( k = 1, 2, 3, \ldots \)) be an event representing \( u_i \) or \( c_{i,j} \). The positive and negative states are denoted by \( e_k^+ \) and \( e_k^- \), respectively. \( u_i \) and \( c_{i,j} \) are elementary events in the sense that their states cannot be defined by any logical expression of states of the other events. \( v_j \) is a composite event, because its positive state can be defined by formula (5). Then, let \( e_k \) be \( e_k^+ \) or \( e_k^- \), and \( \Pi(e_k^+) \) be a possibility equal to a marginal possibility distribution \( \pi(e_k^+) \) on \( E_k = \{ e_k^+, e_k^- \} \). Note that the distributions are defined on \( \{ u_i^+, u_i^- \} \) or \( \{ c_{i,j}^+, c_{i,j}^- \} \) in the paper, but not on \( U \) nor \( C \). Due to formulae (2)–(4), \( \Pi(c_{i,j}^+) \leq \Pi(u_i^+) \), \( \Pi(c_{i,j}^-) \leq \Pi(v_j^+) \), \( \Pi(c_{i,j}^-) \geq \Pi(u_i^-) \) and \( \Pi(c_{i,j}^-) \geq \Pi(v_j^-) \) must be satisfied.

Let \( \Pi(e_1^+ \land e_2^+) \) be a possibility of the joint event \( \langle e_1^+, e_2^+ \rangle \), and be equal to the joint possibility distribution \( \pi(e_1^+, e_2^+) \) on \( E_1 \times E_2 \). Then, possibility \( \Pi(e_1^+) \) is equal to the maximum of \( \Pi(e_1^+ \land e_2^+) \) and \( \Pi(e_1^+ \land e_2^-) \).

\[
\Pi(e_1^+) = \bigvee_{x \in \{ e_2^+, e_2^- \}} \Pi(e_1^+ \land x),
\]

(9)

For conditional possibility, it is assumed that \( \Pi(d|e) \lor \Pi(\bar{d}|e) = 1 \) holds for any states of events \( e \) and \( d \), where \( \bar{d} \) is the negation of \( d \). The relation between joint and conditional possibilities is given by

\[
\Pi(e_1^+ \land e_2^+) = \Pi(e_1^+|e_2^+) \land \Pi(e_2^+),
\]

(10)

which was first introduced in [13].

Suppose that \( \Pi(e_1^+ \land e_2^+) \) and \( \Pi(e_2^+) \) are known and \( \Pi(e_1^+|e_2^+) \) is unknown in Eq. (10). Though the solution is not unique in this case, the paper chooses the largest one as its solution according to the idea of the principle of the maximal specificity or the least specific solution [5,8,10]. In the case of Eq. (10), the principle gives the next solution [13].

\[
\Pi(e_1^+|e_2^+) = \begin{cases} 
\Pi(e_1^+ \land e_2^+), & \text{if } \Pi(e_2^+) > \Pi(e_1^+ \land e_2^+), \\
1, & \text{if } \Pi(e_2^+) = \Pi(e_1^+ \land e_2^+).
\end{cases}
\]

(11)

Then, the next equality is easily obtained using Eq. (11):

\[
\Pi(e_1^+ \land e_2^+|e_3^+) = \Pi(e_1^+|e_2^+ \land e_3^+) \land \Pi(e_2^+|e_3^+).
\]

(12)
Now, if Eq. (13) holds, it is said that $e_2^*$ is *possibilistically independent* of $e_1^*$ [13]. When Eq. (14) is satisfied, $e_1^*$ and $e_2^*$ are *noninteractive* [24]:

$$\Pi(e_2^*|e_1^*) = \Pi(e_2^*),$$  \hfill (13)

$$\Pi(e_1^* \wedge e_2^*) = \Pi(e_1^*) \wedge \Pi(e_2^*).$$  \hfill (14)

$e_1^*$ and $e_2^*$ are noninteractive, if $e_2^*$ ($e_1^*$) is *possibilistically independent* of $e_1^*$ ($e_2^*$).

For the correspondence between the logical and the possibilistic expression, it is defined that positive state $e^+$ is true ($e^−$ is false) iff $\Pi(e^+|e^+) = 1$ and $\Pi(e^−|e^+) = 0$. Iff $\Pi(e^+|e^+)=0$ and $\Pi(e^-|e^+) = 1$, $e^+$ is false ($e^−$ is true). Then, it is easily obtained that $\Pi(e^+|e^+) = \Pi(e^-|e^-) = 1$ from Eq. (11). The equation also gives $\Pi(e^+|e^−) = 0$ if $\Pi(e^−) \neq 0$, and $\Pi(e^+|e^−) = 1$ otherwise. Similarly, $\Pi(e^-|e^+)=0$ holds if $\Pi(e^-) \neq 0$, and $\Pi(e^-|e^+) = 1$ otherwise.

Now, we are ready to define conditional causal possibility. The conditional causal possibilities of $c_{i,j}$ are possibilities of $c_{i,j}^+$ and $c_{i,j}^−$ conditioned by $u_i^+$, and are denoted by $\Pi(c_{i,j}^+|u_i^+)$ and $\Pi(c_{i,j}^-|u_i^+)$, respectively. $\Pi(c_{i,j}^+|u_i^+)$ is the possibility that $u_i^+$ causes $v_j^+$ given $u_i^+$ ($u_i$ causes $v_j$ when presence of $u_i$ is given). $\Pi(c_{i,j}^-|u_i^−)=1$ is easily derived from formulae (3) and (11). $\Pi(c_{i,j}^+|u_i^-)=0$ is derived from formulae (2) and (11) if $\Pi(u_i^-) \neq 0$, while $\Pi(c_{i,j}^+|u_i^-)=1$ if $\Pi(u_i^-) = 0$.

If $\Pi(c_{i,j}^+|u_i^-)>0$ holds, $u_i$ has a causality on $v_j$. Existence of causality from $u_i$ to $v_j$ is denoted by a proposition $\alpha(u_i,v_j)$. From the definition, $\Pi(c_{i,j}^+|u_i^-)=0$ if $\alpha(u_i,v_j)$ is false. Thus, when $\alpha(u_i,v_j)$ is false, $\Pi(c_{i,j}^+|u_i^-)=1$ holds, because $\Pi(c_{i,j}^+|u_i^-) \vee \Pi(c_{i,j}^-|u_i^-) = 1$. In addition, in this case, $\Pi(c_{i,j}^+)=\Pi(c_{i,j}^-|u_i^−) \wedge \Pi(u_i^+) = 0$ and $\Pi(c_{i,j}^-)=\{\Pi(c_{i,j}^−|u_i^−) \wedge \Pi(u_i^+) \} \vee \{\Pi(c_{i,j}^−|u_i^-) \wedge \Pi(u_i^-)\} = \Pi(u_i^+) \vee \Pi(u_i^-)=1$ are derived. Thus $\alpha(u_i,v_j) \rightarrow c_{i,j}^+$, namely, the causation event $c_{i,j}$ never happens if $\alpha(u_i,v_j)$ is false.

Let $\Delta_{i,j}$ be a conjunction of states of any causes and causation events other than the states of the causation event $c_{i,j}$. If the following two equations hold, $c_{i,j}$ ($c_{i,j}^+$ and $c_{i,j}^−$) is *possibilistically causation independent*.

$$\Pi(c_{i,j}^+|u_i^+ \wedge \Delta_{i,j}) = \Pi(c_{i,j}^+|u_i^+),$$  \hfill (15a)

$$\Pi(c_{i,j}^-|u_i^- \wedge \Delta_{i,j}) = \Pi(c_{i,j}^-|u_i^-),$$  \hfill (15b)

where $u_i^+ \wedge \Delta_{i,j}$ must not be contradictory. This means that the values of $\Pi(c_{i,j}^+|u_i^+)$ and $\Pi(c_{i,j}^-|u_i^-)$ are not affected by the states of causes and causation events except for $u_i$ and $c_{i,j}$, when $c_{i,j}$ is possibilistically causation independent.

### 3. Conditional causal possibility by multiple causes

Conditional causal possibility by multiple causes is discussed in this section to deal with compound effects such as synergistic and canceling effects by multiple causes. First, joint-causation events without any compound effects are defined [17,18]. Then, causation events with compound effects are discussed.
**Definition 3.1.** A joint-cause \( u_{g,h} \) is an event that arises when all causes found in the set \( \{u_g, \ldots, u_h\} \) arise, and is defined by its positive state \( u^+_{g,h} \equiv u_g^+ \land \cdots \land u_h^+ \). \( \Phi(u_{g,h}) \) denotes the set of causes composing the joint-cause \( u_{g,h} \), namely \( \Phi(u_{g,h}) = \{u_g, \ldots, u_h\} \). Causes \( u_i \in \Phi(u_{g,h}) \) are called sub-causes of \( u_{g,h} \). Then, if a causation event \( c_{g,h,j} \) with a joint-cause satisfies the next logical formula, it is called joint-causation event.

\[
\begin{align*}
\mathcal{C}^+_g & \iff \bigvee_{i=g}^h \mathcal{C}^+_{i,j} \land u^+_{g,h}.
\end{align*}
\] (16)

A joint-causation event is the one where multiple causes are present and causation events with one of the causes occur disjunctively. The notion of the joint-causation event makes the mechanism of causation clear when multiple causes generate an effect disjunctively.

From the definition, conditional causal possibilities of a joint-causation event are given in the following equations, if all of \( c_{i,j} \) are possibilistically causation independent (see Appendix A):

\[
\begin{align*}
\Pi(c^+_g | u^+_g) & = \bigvee_{i=g}^h \Pi(c^+_i | u^+_i), \quad (17a) \\
\Pi(c^-_g | u^+_g) & = \bigwedge_{i=g}^h \Pi(c^-_i | u^+_i). \quad (17b)
\end{align*}
\]

Then, it is easily proved from the above that the following holds:

\[
\Pi(c^+_g | u^+_g) \lor \Pi(c^-_g | u^+_g) = 1.
\] (18)

If \( \Pi(c^+_g | u^+_g) > 0 \), a proposition \( \mathcal{Z}(u_{g,h}, v_j) \) is true similarly to \( \mathcal{Z}(u_i, v_j) \). \( \mathcal{Z}(u_i, v_j) \rightarrow \mathcal{Z}(u_{g,h}, v_j) \) holds, when \( u_i \) is a sub-cause of \( u_{g,h} \).

A joint-causation event is an event that happens in the case where there is no compound effect by multiple causes. If there is a canceling effect by multiple causes, \( \mathcal{Z}(u_i, v_j) \rightarrow \mathcal{Z}(u_{g,h}, v_j) \) is not necessarily true, even if \( u_i \) is a sub-cause of \( u_{g,h} \).

From now, a new model including compound effects will be discussed.

**Definition 3.2.**

1. A joint-cause \( u_{e,f} \) is a sub-joint-cause of \( u_{g,h} \), iff \( \Phi(u_{e,f}) = \{u_e, \ldots, u_f\} \) is a proper subset of \( \Phi(u_{g,h}) = \{u_g, \ldots, u_h\} \).
2. If a joint-cause \( u_{g,h} \) does not satisfy both Eqs. (17a) and (17b) for an effect \( v_j \), \( u_{g,h} \) has a compound causality on the effect \( v_j \). In this case, \( u_{g,h} \) is said to be compound cause of the effect \( v_j \) and \( c_{g,h,j} \) is compound causation event.
3. A compound causation event \( c_{g,h,j} \) is minimal, iff no sub-joint-cause of \( u_{g,h} \) has a compound causality on \( v_j \). Then, proposition \( \beta(u_{g,h}, v_j) \) is true, iff \( c_{g,h,j} \) is a minimal compound causation event.

The compound causalities defined above do not satisfy Eq. (18) in general. However, the paper sticks to this condition to stay in the formal discussion.
Assumption 3.1. Conditional causal possibilities of any compound causation event $c_{g,h,j}$ satisfy Eq. (18).

Given Assumption 3.1, when Eqs. (17a) and (17b) are not both satisfied at the same time, it is simple to prove that the conditional causal possibilities have necessarily to satisfy one of the following two conditions:

$$
\begin{align*}
\left\{ \Pi(c_{g,h,j}^+|u_{g,h}^+) \geq \bigvee_{i=g,h} \Pi(c_{i,j}^+|u_i^+) \right\} \land \left\{ \Pi(c_{g,h,j}^-|u_{g,h}^-) \leq \bigwedge_{i=g,h} \Pi(c_{i,j}^-|u_i^-) \right\}, \\
\left\{ \Pi(c_{g,h,j}^+|u_{g,h}^+) \leq \bigvee_{i=g,h} \Pi(c_{i,j}^+|u_i^+) \right\} \land \left\{ \Pi(c_{g,h,j}^-|u_{g,h}^-) \geq \bigwedge_{i=g,h} \Pi(c_{i,j}^-|u_i^-) \right\}.
\end{align*}
$$

The compound causalities satisfying formula (19a) are called synergistic, and those satisfying formula (19b) instead are canceling. Note that the case where $\Pi(c_{g,h,j}^+|u_{g,h}^+)=0$ and $\Pi(c_{g,h,j}^-|u_{g,h}^-)=1$ is possible, even if $u_{g,h}$ has a compound causality on $v_j$. Since a compound causation event is also a causation event, formulae from (1) to (4) hold, if $u_i$ is replaced by $u_{g,h}$.

In the rest of the paper, we discuss only minimal compound causalities (but let us call them “compound causalities” for convenience) because general or higher-order compound causalities make the following discussion too complex. They are recalled at the end of the paper and are discussed briefly in an intuitive way. In addition, it is supposed that $u_{g,h}$ always denotes a compound cause of an effect or some effects, because other joint-causes can be handled without using the specific symbols as seen in Eqs. (17a) and (17b). The term single cause is used instead of cause to distinguish it from the compound one.

4. Causal model with compound causalities

Let $U_s = \{u_i|i = 1, \ldots, I\}$ be a set of single causes and $V = \{v_j|j = 1, \ldots, J\}$ be a set of effects. $U_c$ is the set of all compound causes that have (minimal) compound causalities on an effect or some effects. $u_{g,h} \in U_c$ may be referred to as a compound cause without designating its effects caused by $u_{g,h}$, if there is no need to do so. Let $R = U_s \cup U_c$. Elements in $R$ are also denoted by $r_k$ ($k = 1, \ldots, K$, $K = I + |U_c|$), where $|U_c|$ is the cardinality of $U_c$.) for convenience. $r_k$ ($1 \leq k \leq I$) are single causes and $r_k$ ($I < k \leq K$) are compound ones (of an effect or some effects). The notations of $u_i$ and $u_{g,h}$ are also used instead of $r_k$, when single and compound causes must be distinguished. Some pairs of $(u_i, v_j)$ have a causality, and some of $(u_{g,h}, v_j)$ have a compound causality. The mandatory causation assumption is also assumed, though the logical expression must be

$$
v_j^+ \leftrightarrow \bigvee_{r_k \in R} c_{k,j}^+ \leftrightarrow \bigvee_{u_i \in U_s} c_{i,j}^+ \lor \bigvee_{u_{g,h} \in U_c} c_{g,h,j}^+ \quad (20)
$$

instead of formula (5).

A graphical representation of the causal model is shown in Fig. 1. Every node in the network shows a single cause ($u_1, u_2, u_3, u_4$), an effect ($v_1, v_2, v_3$), or a compound cause of an effect
(\(u_{12}, u_{234}\)). In this example, \(U_s = \{u_1, u_2, u_3, u_4\}\), \(V = \{v_1, v_2, v_3\}\) and \(U_c = \{u_{12}, u_{234}\}\). Solid lines with an arrow connecting a single cause \(u_i\) and an effect \(v_j\) are causal links, and show causalities \(\mathcal{A}(u_i, v_j)\). Other solid lines between a compound cause \(u_{g,h}\) and an effect \(v_j\) are also causal links, but show compound causalities \(\mathcal{B}(u_{g,h}, v_j)\). The pairs of numerical values attached to the single causes are possibilities of their states, \((\Pi(u_i^+), \Pi(u_i^-))\). Those attached to the causal links are their conditional causal possibilities, \((\Pi(c_{i,j}^+, r_k^+), \Pi(c_{i,j}^-, r_k^-))\). The link between \(u_{12}\) and \(v_1\) is a canceling causality, and the one between \(u_{234}\) and \(v_3\) is a synergistic causality. The broken lines connecting a single cause and a compound cause are called compound links.

If there is no causal link between a single cause \(u_i\) and an effect \(v_j\), it shows that \(\mathcal{A}(u_i, v_j)\) is false. Thus, the causation event \(c_{i,j}\) never arises for such pairs of \(u_i\) and \(v_j\) as discussed in Section 2. If there is no link between a compound cause \(u_{g,h}\) in \(U_c\) and an effect \(v_j\), it shows that \(\mathcal{B}(u_{g,h}, v_j)\) is false for the pair of \(u_{g,h}\) and \(v_j\), and that the causation between them is defined by formula (16). Thus, it is proved that the next formula holds (see Appendix B).

\[
v_j^+ \leftrightarrow \bigvee_{u_i \in \{u_i|\mathcal{A}(u_i,v_j)\}, u_i \in U_s} c_{i,j}^+ \lor \bigvee_{u_{g,h} \in \{u_{g,h}|\mathcal{B}(u_{g,h},v_j)\}, u_{g,h} \in U_c} c_{g,h,j}^+.
\] (21)

The causal model has the following constraints for the structure:

(a) \(\forall u_{g,h}; |\mathcal{A}(u_{g,h})| \geq 2\),
(b) \(\forall u_{g,h}, \exists v_j; \mathcal{B}(u_{g,h}, v_j)\),
(c) \(\forall u_i, \exists v_j; \mathcal{A}(u_i, v_j) \lor \{u_i \in \mathcal{A}(u_{g,h}) \land \mathcal{B}(u_{g,h}, v_j)\}\),
(d) \(\forall v_j, \exists u_i, \exists u_{g,h}; \mathcal{A}(u_i, v_j) \lor \mathcal{B}(u_{g,h}, v_j)\).

Condition (a) means that a compound cause has at least two compound links. (b) shows that a compound cause has at least a causal link expressing a compound causality. (c) says that a single cause has at least a causal link or a compound link. Then, (d) means that an effect has at least a causal link.
Now, the definitions of conditional causal possibility and possibilistic causation independence should be revised, since compound causation events are newly introduced.

**Definition 4.1.**

(1) $\Pi(c_{k,j}^+ | r_k^+)$ and $\Pi(c_{k,j}^- | r_k^-)$ $(k = 1, \ldots, K, K = |U_s| + |U_c|)$ are called conditional causal possibilities of $c_{k,j}$.

(2) Let $\Delta_{i,j}$ be a conjunction of states of any (single and compound) causes and causation events other than the states of the causation event $c_{i,j}$ $(1 \leq i \leq I = |U_i|)$. If there is no compound cause $u_{g,h}$ that satisfies the next formula in $U_s$, $\Delta_{i,j}$ is said to be a context of $c_{i,j}$:

$$
\beta(u_{g,h}, v_j) \wedge \{ u_i \in \Phi(u_{g,h}) \} \wedge \{ (\Delta_{i,j} \wedge u_{k}^+) \to u_{g,h}^+ \}.
$$

(3) Let $\Delta_{k,j}$ be a conjunction of states of any causes and causation events other than the states of the compound causation event $c_{k,j}$ $(1 < k \leq K)$. Then, $\Delta_{k,j}$ is a context of $c_{k,j}$.

(4) Let $\Delta_{k,j}$ be a context of $c_{k,j}$ $(k = 1, \ldots, K)$. $r_k^+ \wedge \Delta_{k,j}$ must not be contradictory. Then, iff $\Pi(c_{k,j}^+ | r_k^+ \wedge \Delta_{k,j}) = \Pi(c_{k,j}^+ | r_k^+)$ and $\Pi(c_{k,j}^- | r_k^+ \wedge \Delta_{k,j}) = \Pi(c_{k,j}^- | r_k^+)$ hold for any $\Delta_{k,j}$, $c_{k,j}$ is possibilistically causation independent in the compound causal model.

Definition 4.1(2) means that $\Delta_{i,j}$ $(1 \leq i \leq I)$ is a context of $c_{i,j}$, iff there is no compound cause $u_{g,h}$ that has a compound causality on $v_j$, that includes $u_i$ as its sub-cause, and whose positive state $u_{g,h}^+$ is logically implied by $\Delta_{i,j} \wedge u_{i}^+$. On the other hand in Definition 4.1(3), $\Delta_{k,j}$ $(1 < k \leq K)$ is a context of $c_{k,j}$ without such conditions. Possibilistic causation independence in the compound causal model is defined using these contexts. (Hereafter, the term “possibilistically causation independent” is used in the meaning of this definition.) The following example illustrates the complex definitions.

**Example 4.1.** In the case of causal model in Fig. 1, $c_{1,1}^+ \wedge u_{1}^+, u_{1}^+ \wedge u_{3}^+$ and $u_{2}^+ \wedge u_{3}^-$ are all contexts of $c_{4,3}$, because there is no compound cause that satisfies formula (22). Thus, if $c_{4,3}$ is possibilistically causation independent in the sense of Definition 4.1, we get $\Pi(c_{4,3}^+ | c_{1,1}^+ \wedge u_{2}^+ \wedge u_{3}^+) = \Pi(c_{4,3}^+ | u_{1}^+ \wedge u_{3}^+ \wedge u_{3}^-) = \Pi(c_{4,3}^+ | u_{3}^+)$.

However, $c_{2,1}^+ \wedge u_{3}^-$ is not a context of $c_{4,3}$, because $\beta(u_{234}, v_3) \wedge \{ u_4 \in \Phi(u_{234}) \} \wedge \{ (c_{2,1}^+ \wedge u_{3}^+ \wedge u_{3}^-) \to u_{234}^- \}$. Thus, even if $c_{4,3}$ is possibilistically causation independent, $\Pi(c_{4,3}^+ | c_{2,1}^+ \wedge u_{3}^+ \wedge u_{3}^-) \neq \Pi(c_{4,3}^+ | u_{3}^+)$. Similarly, $u_{2}$ is not a context of $c_{1,1}$, because $\beta(u_{12}, v_1) \wedge \{ u_1 \in \Phi(u_{12}) \} \wedge \{ (u_{1}^+ \wedge u_{2}^+) \to u_{12}^+ \}$. Therefore, $\Pi(c_{1,1}^+ | u_{1}^+ \wedge u_{2}^+) \neq \Pi(c_{1,1}^+ | u_{1}^-)$.

These examples show that it is possible to have $\Pi(c_{i,j}^+ | u_{i}^+ \wedge \Delta_{i,j}) \neq \Pi(c_{i,j}^+ | u_{j}^+)$ even if $c_{i,j}$ is possibilistically causation independent, when $u_{i}^+ \wedge \Delta_{i,j}$ implies the positive state of a compound cause $u_{g,h}$ that has $u_i$ as a sub-cause and has a compound causality on $v_j$.

Now, the following is assumed.

**Assumption 4.1.**

(1) Any $u_{i}^+$ $(1 \leq i \leq I)$ is possibilistically independent of $u_{i'}^+$ and $u_{i}^-$ $(1 \leq i' \leq I, i' \neq i)$. In the same way, any $u_{i}^-$ is possibilistically independent of $u_{i'}^+$ and $u_{i}^-$.

(2) Any causation event $c_{k,j}$ $(j = 1, \ldots, J, k = 1, \ldots, K)$ is possibilistically causation independent.
Note that possibilistic independence among the states of \( r_k \) \((1 \leq k \leq K)\) does not hold in general. From the definition of \( u^+_{g,h} \) and Assumption 4.1(1), possibilities of \( u^+_{g,h} \) and \( u^-_{g,h} \) are given by the following equations:

\[
\begin{align*}
\Pi(u^+_{g,h}) &= \bigwedge_{i=g,...,h} \Pi(u^+_i) \tag{23a}, \\
\Pi(u^-_{g,h}) &= \bigvee_{i=g,...,h} \Pi(u^-_i). \tag{23b}
\end{align*}
\]

Assumption 4.1(2) gives us a means to calculate the values of \( \Pi(c^+_{k,j} \mid r^+_k \land \Delta_{k,j}) \) and \( \Pi(c^-_{k,j} \mid r^+_k \land \Delta_{k,j}) \) when \( \Delta_{k,j} \) is a context of \( c_{k,j} \). However, one more assumption is necessary to cope with the case where \( \Delta_{i,j} \) is not a context of \( c_{i,j} \).

**Assumption 4.2.** Let \( \Delta_{i,j} \) be a conjunction of states of any causes and causation events other than the states of the causation event \( c_{i,j} \) \((1 \leq i \leq I)\). If \( \Delta_{i,j} \) is not a context of \( c_{i,j} \), then \( \Pi(c^+_{i,j} \mid u^+_i \land \Delta_{i,j}) = 0.0 \) and \( \Pi(c^-_{i,j} \mid u^-_i \land \Delta_{i,j}) = 1.0 \).

In the case of Assumption 4.2, there is at least a compound cause \( u^+_{g,h} \) satisfying formula (22), and some \( \mu \) could be defined so that formula \( u^+_i \land \Delta_{i,j} \leftrightarrow u^+_{g,h} \land \mu \) would be satisfied. Then, the assumption says that \( \Pi(c^+_{i,j} \mid u^+_{g,h} \land \mu) = 0 \) and \( \Pi(c^-_{i,j} \mid u^+_{g,h} \land \mu) = 1 \) hold even if \( \Pi(c^+_{i,j} \mid u^+_i) > 0 \) or \( \Pi(c^-_{i,j} \mid u^-_i) < 1 \). In short, the assumption means that causal links between \( u_i \) and \( v_j \) should be neglected, when (1) \( u_i \) is a sub-cause of a compound cause \( u^+_{g,h} \), (2) the compound cause has a compound causality on \( v_j \), and (3) the positive state of the compound cause is given. Using the assumption, the values of \( \Pi(c^+_{i,j} \mid u^+_i \land \Delta_{i,j}) \) and \( \Pi(c^-_{i,j} \mid u^-_i \land \Delta_{i,j}) \) can be calculated even if \( \Delta_{i,j} \) is not a context of \( c_{i,j} \) as shown in the following example.

**Example 4.2.** In the model of Fig. 1, the possibility of \( v^+_1 \) given \( u^+_1 \), \( u^+_2 \) and \( u^+_3 \) is calculated using formula (21) as

\[
\begin{align*}
\Pi(v^+_1 \mid u^+_1 \land u^+_2 \land u^+_3) &= \Pi(c^+_{1,1} \lor c^+_{2,1} \lor c^+_{12,1} \mid u^+_1 \land u^+_2 \land u^+_3) \\
&= \Pi(c^+_{1,1} \mid u^+_1 \land u^+_2 \land u^+_3) \lor \Pi(c^+_{2,1} \mid u^+_1 \land u^+_2 \land u^+_3) \lor \Pi(c^+_{12,1} \mid u^+_1 \land u^+_2 \land u^+_3).
\end{align*}
\]

If \( u^+_2 \land u^+_3 \) was a context of \( c_{1,1} \) and \( u^+_1 \land u^+_3 \) was a context of \( c_{2,1} \), the possibility would be calculated as follows:

\[
\Pi(v^+_1 \mid u^+_1 \land u^+_2 \land u^+_3) = \Pi(c^+_{1,1} \mid u^+_1 \land u^+_3) \lor \Pi(c^+_{2,1} \mid u^+_2 \land u^+_3) \lor \Pi(c^+_{12,1} \mid u^+_1 \land u^+_2 \land u^+_3) = 0.8.
\]

This result is inexcusable, because the canceling causality by \( u_{12} \) does not play the role. In reality, neither \( u^+_2 \land u^+_3 \) nor \( u^+_1 \land u^+_3 \) are contexts of \( c_{1,1} \) and \( c_{2,1} \) respectively, though \( u^+_2 \) is a context of \( c_{12,1} \). Thus, we get \( \Pi(c^+_{1,1} \mid u^+_1 \land u^+_2 \land u^+_3) = 0 \) and \( \Pi(c^+_{2,1} \mid u^+_1 \land u^+_2 \land u^+_3) = 0 \) from Assumption 4.2, and \( \Pi(c^+_{12,1} \mid u^+_1 \land u^+_2 \land u^+_3) = \Pi(c^+_{12,1} \mid u^+_1) = 0.4 \) from Assumption 4.1(2). Thus, the possibility is obtained as \( \Pi(v^+_1 \mid u^+_1 \land u^+_2 \land u^+_3) = \max(0,0,0.4) = 0.4 \).
As discussed above, the calculation of the possibility of an effect state conditioned by a set of states of causes must take into account the relations among the causalities. Now, the next definition is introduced to make the problem clear.

**Definition 4.2.** Let us suppose \( u_{g,h} \) has a compound causality on \( v_j \), i.e. \( \beta(u_{g,h}, v_j) \) is true. If there exists a cause \( u_i \) in \( \Phi(u_{g,h}) \) that has a causality on the effect \( v_j \), the pair \((u_{g,h}, v_j)\) is said to be multiply-connected, and singly-connected otherwise. Formally, \((u_{g,h}, v_j)\) is multiply-connected if the following condition is satisfied:

\[
\exists u_i \in \Phi(u_{g,h}); \alpha(u_i, v_j) \land \beta(u_{g,h}, v_j). \tag{24}
\]

In Example 4.2, \( u_{12} \) and \( v_1 \) are multiply-connected and \( u^+_{12} \) is given. In this case, Assumption 4.2 plays an important role as seen in the example. On the other hand, attention to the assumption is unnecessary when they are singly-connected. If \( u_{g,h} \) and \( v_j \) are singly-connected, formula (24) is denied. This means that \( \beta(u_{g,h}, v_j) \rightarrow \alpha(u_i, v_j) \) and \( \alpha(u_i, v_j) \rightarrow \beta(u_{g,h}, v_j) \) hold for \( \forall u_i \in \Phi(u_{g,h}) \). Thus, if \( \beta(u_{g,h}, v_j) \) is true, it is guaranteed that any sub-cause of \( u_{g,h} \) never generates \( v_j \). If \( \alpha(u_i, v_j) \) is true, there is no compound cause that has \( u_i \) as its sub-cause and a compound causality on \( v_j \).

5. Compound causality consistency problem

Let us suppose that a causal model with compound causalities is developed to analyze a certain system. Possibilities attached to the model are our knowledge about uncertainty obtained from experiences in the past. They could be called prior possibilities in the same sense as the prior probabilities. Then, suppose that states of some causes and/or effects in the model are given or observed in a certain situation, for example by inspecting the original system of the model. The given observation is represented by a conjunction of states of causes and effects, for example, by \( u^+_1 \land u^-_2 \land v^-_3 \land v^+_3 \). (This observation is logically equivalent to \( u^+_1 \land u^-_2 \land v^-_3 \land v^+_3 \), because \( u^-_{34} \leftrightarrow u^-_3 \lor u^-_4 \). However, \( u^-_{34} \) is included in the observation, because compound causes play a crucial role in this problem.) The problem addressed here is one to calculate the posterior possibilities, if we could say so, of a hypothesis about some unobserved causes and effects given the observation. The hypothesis is also given by a conjunction of states of causes and effects, for example, by \( u^+_2 \land v^-_1 \land u^-_{12} \). (This hypothesis is logically equivalent to \( u^+_1 \land u^+_2 \land v^-_1 \land u^-_{12} \). However, \( u^+_1 \) is excluded from the hypothesis, because it is included in the observation and not a hypothesis from the viewpoint of the problem solving.)

The posterior possibilities of a hypothesis are defined by \( \Pi_{\text{post}}(\text{hypothesis}) = \Pi(\text{hypothesis} \mid \text{observation}) \) and \( \Pi_{\text{post}}(\text{hypothesis}) = \Pi(\text{hypothesis} \mid \text{observation}) \), where hypothesis and observation are conjunctions of states of causes and effects. If the above examples of observation and hypothesis are used, possibilities to be obtained in the problem are \( \Pi_{\text{post}}(u^+_2 \land v^-_1 \land u^-_{12}) = \Pi(u^+_2 \land v^-_1 \land u^-_{12} \mid u^+_1 \land u^-_2 \land v^-_1 \land u^-_{12} \land u^-_{34}) \) and \( \Pi_{\text{post}}(u^+_2 \land v^-_1 \land u^-_{12}) = \Pi(u^+_2 \land v^-_1 \land u^-_{12} \mid u^+_1 \land u^-_3 \land v^-_2 \land v^-_{12} \land u^-_{34}) \).

Let us define the problem in a general way. Let \( U^*_s \subseteq U_s = \{u_1, \ldots, u_I\} \) and \( V^*_s \subseteq V = \{v_1, \ldots, v_J\} \) be sets of single causes and effects whose states are observed, respectively \((U^*_s = \{u_1, u_3\}) \) and \( V^*_s = \{v_2, v_3\} \) in the above example. The observed states of \( u_i \) and \( v_j \) are denoted by \( u^*_i \) (or \( r^*_k \)) and \( v^*_j \) respectively, though the actual states are \( u^-_i \) or \( u^+_i \), and \( v^-_j \) or \( v^+_j \). The sets of the observed states \( u^*_i \) and \( v^*_j \) are denoted by \( P^*_s \) and \( Q^* \), respectively \((P^*_s = \{u^+_1, u^-_3\}) \) and \( Q^* = \{v^-_2, v^-_3\} \).
$P_s^* \cup Q^* \neq \emptyset$ is assumed. The states of compound causes in $U_c$ can be calculated logically from those of their sub-causes, because $u_{g,h}^+ \leftrightarrow u_g^+ \land \cdots \land u_h^+$. Thus, states of some compound causes may be given by $P_s^*$ (if $u_{34}$ is in $U_c$, its state $u_{34}^-$ is derived from $u_3^-$ in $P_s^*$ because $u_3^- \rightarrow u_{34}^-$. Let $P_c^*$ be the set of the states of such compound causes derived from $P_s^*$ ($P_c^* = \{u_{34}\}$). Formally, $P_c^* = \{u_{g,h}^+ \mid \bigwedge_{u_i \in P_c} u_i^+ \rightarrow u_{g,h}^+, u_{g,h} \in U_c\}$, where $u_{g,h}^+ = u_g^+ \land u_h^+$. Let $P^* = P_s^* \cup P_c^* (P^* = \{u_1^+, u_3^+, u_{34}\})$. Then, the conjunction of elements in $P^* \cup Q^*$ gives the observation $(u_1^+ \land u_3^- \land v_2^- \land v_3^+ \land u_{34}^-)$. Then, let $\hat{U}_s \subseteq U_s - U_s^*$ and $\hat{V} \subseteq V - V^*$ be the sets of single causes and effects whose states are hypothesized ($\hat{U}_s = \{u_2\}, \hat{V} = \{v_1\}$ in the above example). The hypothesized states are denoted by $\hat{u}_i (\hat{r}_k)$ and $\hat{v}_j$, where $\hat{u}_i$ is $u_i^+$ or $u_i^-$, and $\hat{v}_j$ is $v_j^+$ or $v_j^-$. The sets of $\hat{u}_i$ and $\hat{v}_j$ are $\hat{P}_s$ and $\hat{Q}$, respectively ($\hat{P}_s = \{u_2^+\}, \hat{Q} = \{v_1^-\}$). $\hat{P}_s \cup \hat{Q} \neq \emptyset$ is assumed. Furthermore, let $\tilde{P}_c$ be the set of states of compound causes that can be determined using the hypothesized states in $\tilde{P}_s$, namely determined from $\tilde{P}_s$ and both from $P_s^*$ and $\hat{P}_s$, but not only from $P_s^*$ (if $u_{12}$ is in $U_c$, its state $u_{12}^-$ is derived from both $P_s^* = \{u_1^+, u_3^-\}$ and $P_c^* = \{u_{12}^+\}$, because $u_{12}^- \leftrightarrow u_1^+ \land u_3^- \land u_{12}^- \rightarrow u_{12}^- \land u_{12}^- \rightarrow u_{56}$. So, $\tilde{P}_c = \{u_{12}^-\}$, $u_{34}^-$ is not included in $\tilde{P}_c$, because it is derived only from $P_s^*$. Formally, $\tilde{P}_c = \{\hat{u}_{g,h} \mid \bigwedge_{u_i \in \hat{P}_s \cup \hat{P}_s^*} \hat{u}_i \rightarrow \hat{u}_{g,h}, u_{g,h} \in U_c\} - P_c^*$, where $\hat{u}_{g,h}^+ = u_{g,h}^+, \hat{u}_{g,h}^- = u_{g,h}^-$. $\hat{P}_c$ is given by $\hat{P} \cup \hat{Q} \cup \tilde{P}_c (\hat{P} = \{u_2^+, u_{12}^-\})$. Then, the conjunction of elements in $\hat{P} \cup \hat{Q}$ gives the hypothesis $(u_2^+ \land v_1^+ \land u_{12}^-)$. Now, the compound causality consistency (CCC) problem is to calculate $\text{Pos}(\hat{P}, \hat{Q} \mid P^*, Q^*)$ and $\text{Pos}(\hat{P}, \hat{Q} \mid P^*, Q^*)$ defined by the following equations:

$$
\text{Pos}(\hat{P}, \hat{Q} \mid P^*, Q^*) = \Pi \left( \bigwedge_{\hat{r}_k \in \hat{P}} \hat{r}_k \land \bigwedge_{\hat{v}_j \in \hat{Q}} \hat{v}_j \bigg| \bigwedge_{r_i^* \in P^*} r_i^* \land \bigwedge_{v_j^* \in Q^*} v_j^* \right), 
$$

$$
\text{Pos}(\hat{P}, \hat{Q} \mid P^*, Q^*) = \Pi \left( \bigwedge_{\hat{r}_k \in \hat{P}} \hat{r}_k \land \bigwedge_{\hat{v}_j \in \hat{Q}} \hat{v}_j \bigg| \bigwedge_{r_i^* \in P^*} r_i^* \land \bigwedge_{v_j^* \in Q^*} v_j^* \right),
$$

where $\Pi(u_1^+), \Pi(u_3^-), \Pi(c_{v_1}^+ \mid r_k^+) \Pi(c_{v_2}^- \mid r_k^+)$ and $\Pi(c_{v_3}^- \mid r_k^+)$ are given as a priori knowledge.

An example is shown in Fig. 2. $P^* = \{u_1^+, u_3^+, u_6^-\}$ and $Q^* = \{v_2^+, v_3^-\}$ are sets of the observed states of single causes and effects, respectively. $P_c^* = \{u_{12}^+, u_{36}^+\}$ is the set of states of compound causes derived from $P_s^* = \{u_1^+, u_3^-\}$ and $Q_c^* = \{v_1^+, v_3^+, v_4^+\}$. $\tilde{P}_c = \{u_{34}^+, u_{56}^+\}$. $P^* = P_s^* \cup P_c^* = \{u_1^+, u_3^-, u_6^-, u_{12}^+\}$ and $\hat{P} = \hat{P}_s \cup \tilde{P}_c = \{u_2^+, u_3^+, u_{34}^+, u_{56}^+\}$.

The CCC problem is the one to calculate the values of $\text{Pos}(\{u_3^+, u_4^+, u_{34}^+, v_1^+, v_2^+, v_4^+\} \mid \{u_1^+, u_2^+, u_6^-, u_{12}^+, u_{56}^+\}, \{v_2^+, v_3^-\})$ and $\text{Pos}(\{u_2^+, u_4^+, u_{34}^+, v_1^+, v_2^+, v_4^+\} \mid \{u_1^+, u_2^+, u_6^-, u_{12}^+, u_{56}^+\}, \{v_2^+, v_3^-\})$.

By the way, Eq. (25b) is transformed into the next equation:

$$
\text{Pos}(\hat{P}, \hat{Q} \mid P^*, Q^*) = \Pi \left( \bigvee_{\hat{r}_k \in \hat{P}} \bigwedge_{r_i^* \in P^*} r_i^* \land \bigwedge_{v_j^* \in Q^*} v_j^* \right) = \bigvee_{\hat{r}_k \in \hat{P}} \Pi \left( \bigwedge_{r_i^* \in P^*} r_i^* \land \bigwedge_{v_j^* \in Q^*} v_j^* \right) \bigwedge_{\hat{v}_j \in \hat{Q}} \Pi \left( \bigwedge_{r_i^* \in P^*} r_i^* \land \bigwedge_{v_j^* \in Q^*} v_j^* \right). 
$$
Note that each term of the right hand side of the above equation is a special case of Eq. (25a). Thus, the paper discusses only the way to calculate the value of Eq. (25a).

5.1. Simplification of CCC problem

Let us call elements in $P^*$ and $Q^*$ the observed states. The hypothesized states are those in $\hat{P}$ and $\hat{Q}$. Both the observed and the hypothesized states may be called the assigned states.

When the assigned state of a compound cause in $P^*_c$ or $\hat{P}_c$ is positive (for example, the assigned states of $u_{12}$ and $u_{34}$ are positive in Fig. 2), those of its sub-causes should be also positive. If a positive compound cause has a causal link with an effect and some of its sub-causes also have a causal link with the effect, the links between the sub-causes and the effect must be ignored because of Assumption 4.2. This means that the network can be simplified by removing the causal links.

On the other hand, when the assigned state of a compound cause is negative (the assigned state of $u_{56}$ is negative in Fig. 2), there is no need to consider the cause from the viewpoint of causality. Thus we can remove the compound cause and causal links departing from it. Furthermore, if there are any nodes connecting only to the removed nodes, they can be removed, since they have no influence on other nodes.

In the example of Fig. 2, links and nodes marked by $\times$ can be removed. If a compound cause has an assigned state, both the compound cause and its sub-causes do not have a causal link with the same effect after the simplification. However, if a compound cause has no assigned state, it is not the case. In the example, $u_{45}, u_4$, and $u_5$ have a causal link with $v_4$.

In the rest of the paper, it is assumed that the problem has been simplified in the above way and that all nodes are reordered after the simplification (see Fig. 3).

Now, the next definition is introduced to divide the simplified problems into two classes.
**Definition 5.1.** If there is at least a compound cause that is multiply-connected with an effect, the CCC problem is multiply-connected, otherwise, it is singly-connected.

The simplified problem shown in Fig. 3 is multiply-connected, because $u_{45}$ and $v_4$ are multiply-connected.

### 5.2. Calculation of possibilities

Now, several lemmas and propositions are necessary to solve the CCC problem.

**Lemma 5.1.** Let $r^*_k$ be $r^+_k$ or $r^-_k$ ($k = 1, \ldots, K$), and $c^*_{k,j}$ ($j = 1, \ldots, J$) be $c^+_{k,j}$ or $c^-_{k,j}$. $\omega$ is a conjunction of some $r^*_k$, or else a tautology. Then, $c^*_{k_1,j_1}$ and $c^*_{k_2,j_2}$ ($1 \leq k_1, k_2 \leq K$, $1 \leq j_1, j_2 \leq J$) are noninteractive when conditioned by $\omega$, if $c^*_{k_1,j_1} \land c^*_{k_2,j_2}$ is not contradictory and the CCC problem is singly-connected.

\[
\Pi(c^*_{k_1,j_1} \land c^*_{k_2,j_2} | \omega) = \Pi(c^*_{k_1,j_1} | \omega) \land \Pi(c^*_{k_2,j_2} | \omega),
\]

where \( \Pi(\bullet | \omega) = \Pi(\bullet) \) if $\omega$ is a tautology.

**Proof.** See Appendix C. □

**Lemma 5.2.** Let $r^*_k$ be $r^+_k$ or $r^-_k$ ($k = 1, \ldots, K$), and $c^*_{k,j}$ ($j = 1, \ldots, J$) be $c^+_{k,j}$ or $c^-_{k,j}$. $\omega$ is a conjunction of some $r^*_k$, or else a tautology. $\Pi(\omega) \neq 0$ is assumed. Then, if the CCC problem is
singly-connected, the next equation holds:

$$
\Pi(c_{k,j}^* | \omega) = \bigvee_{w_k \in \{r_k^+, r_k^-\}} \{ \Pi(c_{k,j}^* | w_k) \land \Pi(w_k | \omega) \}. 
$$

(28)

**Proof.** See Appendix D.  □

**Proposition 5.1.** Let \( r_k^* \) be \( r_k^+ \) or \( r_k^- \) (\( k = 1, \ldots, K \)). \( \omega \) is a conjunction of some \( r_k^* \), or else a tautology. \( \Pi(\omega) \neq 0 \) is assumed. Then, if the CCC problem is singly-connected, the next equations hold:

$$
\Pi(v_j^+ | \omega) = \bigvee_{r_k \in \{r_k | x(r_k, \omega) \land \beta(r_k, \omega)\}} \{ \Pi(c_{k,j}^+ | r_k^+) \land \Pi(r_k^+ | \omega) \},
$$

(29a)

$$
\Pi(v_j^- | \omega) = \bigwedge_{r_k \in \{r_k | x(r_k, \omega) \land \beta(r_k, \omega)\}} \bigvee_{w_k \in \{r_k^+, r_k^->\}} \{ \Pi(c_{k,j}^- | w_k) \land \Pi(w_k | \omega) \}. 
$$

(29b)

**Proof.** See Appendix E.  □

**Proposition 5.2.** Let \( r_k^* \) be \( r_k^+ \) or \( r_k^- \) (\( k = 1, \ldots, K \)), and \( v_j^* \) (\( j = 1, \ldots, J \)) be \( v_j^+ \) or \( v_j^- \). \( \omega \) is a conjunction of some \( r_k^* \), or else a tautology. If the CCC problem is singly-connected and \( v_{j_1}^* \land v_{j_2}^* \) is not contradictory, \( v_{j_1}^* \) and \( v_{j_2}^* \) (\( 1 \leq j_1, j_2 \leq J \)) are noninteractive when conditioned by \( \omega \):

$$
\Pi(v_{j_1}^* \land v_{j_2}^* | \omega) = \Pi(v_{j_1}^* | \omega) \land \Pi(v_{j_2}^* | \omega). 
$$

(30)

**Proof.** See Appendix F.  □

Now, to calculate the values of \( \Pi(r_k^+ | \omega) \) and \( \Pi(r_k^- | \omega) \) in Eqs. \( \text{(29a)} \) and \( \text{(29b)} \), the next proposition is used.

**Proposition 5.3.** Let \( r_k^* \) be \( r_k^+ \) or \( r_k^- \) (\( k = 1, \ldots, K \)). \( \omega \) is a conjunction of some \( r_k^* \), or else a tautology. Then, the following equations hold:

$$
\Pi(r_k^+ | \omega) = \bigwedge_{u_j \in \Phi(r_k)} \Pi(u_j^+ | \omega),
$$

(31a)

$$
\Pi(r_k^- | \omega) = \bigvee_{u_j \in \Phi(r_k)} \Pi(u_j^- | \omega).
$$

(31b)

**Proof.** See Appendix G.  □
The value of $\Pi(u_i^+ | \omega)$ in the right-hand side of Eq. (31a) is obtained in the following way:

$$
\Pi(u_i^+ | \omega) = \begin{cases} 
1, & \text{if } \omega \rightarrow u_i^+, \\
0, & \text{if } \omega \rightarrow u_i, \\
\Pi(u_i^+), & \text{otherwise}.
\end{cases}
$$

(32)

$\Pi(u_i^- | \omega)$ is also calculated in the same way.

**Lemma 5.3.** The next equations hold:

- $\Pi(r_{k_1}^+ \land r_{k_2}^+) = \Pi(r_{k_1}^+) \land \Pi(r_{k_2}^+)$,
- $\Pi(r_{k_1}^- \land r_{k_2}^-) = \Pi(r_{k_1}^-) \land \Pi(r_{k_2}^-)$,
- $\Pi(r_{k_1}^+ \land r_{k_2}^-) = \Pi(r_{k_1}^+) \land \Pi(r_{k_2}^- \setminus k_1)$,

where

$$
\Pi(r_{k_2}^- \setminus k_1) = \bigvee_{u_i \in \Phi(r_{k_2}) - \Phi(r_{k_1})} \Pi(u_i^-).
$$

In Eq. (33c), it is assumed that $r_{k_1}^+ \land r_{k_2}^-$ is not contradictory, and $\Phi(r_{k_2}) - \Phi(r_{k_1}) \neq \emptyset$.

**Proof.** See Appendix H. \(\square\)

**Proposition 5.4.** Let $R^p$ be a set of $r_k$ whose states are positive, and $R^n$ be a set of $r_k$ whose states are negative ($k = 1, \ldots, K$). Then, the next equations hold, if the logical formula in the left-hand side is not contradictory:

$$
\Pi \left( \bigwedge_{r_k \in R^p} r_k^+ \land \bigwedge_{r_k \in R^n} r_k^- \right) = \bigwedge_{r_k \in R^p} \Pi(r_k^+) \land \bigwedge_{r_k \in R^n} \Pi(r_k^- \setminus R^p),
$$

(34)

where

$$
r_{h \setminus R^p}^- \leftrightarrow \bigvee_{u_i \in \Phi(r_h) - \Phi(R^p)} u_i^-,
$$

$$
\Phi(R^p) = \bigcup_{r_k \in R^p} \Phi(r_k).
$$

In the above, $\Phi(r_h) - \Phi(R^p) \neq \emptyset$.

**Proof.** See Appendix I. \(\square\)
5.3. **Solutions of the problem**

We are ready to solve the CCC problem.

(1) **The singly-connected problem**

The singly-connected problem is discussed first. Let us define $Pos(P^*, Q^*)$ and $Pos(P^*, Q^*, \hat{P}, \hat{Q})$ as follows:

$$Pos(P^*, Q^*) = \Pi \left( \bigwedge_{r_i^* \in P^*} r_i^* \land \bigwedge_{v_j^* \in Q^*} v_j^* \right) = \Pi \left( \bigwedge_{v_j^* \in Q^*} v_j^* \bigg| \bigwedge_{r_i^* \in P^*} r_i^* \right) \wedge \Pi \left( \bigwedge_{r_i^* \in P^*} r_i^* \right), \quad (35)$$

$$Pos(P^*, Q^*, \hat{P}, \hat{Q}) = \Pi \left( \bigwedge_{v_j^* \in Q^*} v_j^* \land \bigwedge_{\hat{v}_j \in \hat{Q}} \hat{v}_j \land \bigwedge_{r_k^* \in P^*} r_k^* \land \bigwedge_{\hat{r}_k \in \hat{P}} \hat{r}_k \right)$$

$$= \Pi \left( \bigwedge_{v_j^* \in Q^*} v_j^* \bigg| \bigwedge_{\hat{v}_j \in \hat{Q}} \hat{v}_j \bigg| \bigwedge_{r_k^* \in P^*} r_k^* \bigg| \bigwedge_{\hat{r}_k \in \hat{P}} \hat{r}_k \right) \wedge \Pi \left( \bigwedge_{r_k^* \in P^*} r_k^* \right) \wedge \Pi \left( \bigwedge_{\hat{r}_k \in \hat{P}} \hat{r}_k \right) \wedge \Pi \left( \bigwedge_{r_i^* \in P^*} r_i^* \right) \wedge \Pi \left( \bigwedge_{\hat{r}_i \in \hat{P}} \hat{r}_i \right) \quad (36)$$

In Eq. (35) and (36), Proposition 5.2 is used as well as Eq. (10). The values of $Pos(P^*, Q^*)$ and $Pos(P^*, Q^*, \hat{P}, \hat{Q})$ can be calculated using Propositions 5.1, 5.3 and 5.4.

Then, the next equation holds, because Eq. (10) is assumed in the paper:

$$Pos(\hat{P}, \hat{Q}|P^*, Q^*) \land Pos(P^*, Q^*) = Pos(P^*, Q^*, \hat{P}, \hat{Q}). \quad (37)$$

Applying the idea of the least specific solution to the above, the solution of the CCC problem is obtained as follows:

$$Pos(\hat{P}, \hat{Q}|P^*, Q^*) = \begin{cases} Pos(P^*, Q^*, \hat{P}, \hat{Q}), & \text{if } Pos(P^*, Q^*) > Pos(P^*, Q^*, \hat{P}, \hat{Q}), \\ 1, & \text{if } Pos(P^*, Q^*) = Pos(P^*, Q^*, \hat{P}, \hat{Q}). \end{cases} \quad (38)$$

(2) **The multiply-connected problem**

Propositions 5.1 and 5.2 cannot be applied to multiply-connected problems. Then, an approach similar to the one used in Bayesian networks, which converts a multiply-connected network to the singly-connected by fixing states of some events whose states are not assigned [15,16], is introduced. The procedure is as follows: (1) a possible state is tentatively assigned to some single causes so
that states of all compound causes could be assigned (let  $\hat{S}$ be a set of tentatively assigned states of the single and compound causes). (2) Then, the multiply-connected problem can be converted to a singly-connected by re-simplifying it. (3) The possibilities of $\text{Pos}(\hat{P} \cup \hat{S} | \hat{Q} | P^*, Q^*)$ are calculated for all combinations of  $\hat{S}$. (4) The value of $\text{Pos}(\hat{P}, \hat{Q} | P^*, Q^*)$ is obtained as the maximum among these possibilities.

The above procedure could be formalized in this way. Let $U_s^*, \hat{U}_s, U_c^*, \hat{U}_c$ be sets of causes whose states are given by $P^*, \hat{P}, P_c^*, \hat{P}_c$, respectively. If the simplified CCC problem is multiply-connected, there is at least a compound cause in $U_c^\circ = U_c - (U_c^* \cup \hat{U}_c)$ and their states are not assigned. Then, $\text{Pos}(\hat{P}, \hat{Q} | P^*, Q^*)$ is given by

$$\text{Pos}(\hat{P}, \hat{Q} | P^*, Q^*) = \bigvee_{x_i \in \{u_i^+, u_i^-\}, u_i \in U_i^\circ} \Pi \left( \bigwedge_{u_i \in U_i^\circ} x_i \land \bigwedge_{r_h \in U_h^\circ} \tilde{r}_h \land \bigwedge_{\tilde{r}_j \in \hat{Q}} \tilde{r}_j \land \bigwedge_{v_j \in Q^*} v_j \right),$$

where

$$U_i^\circ = \bigcup_{r_h \in U_h^\circ} \Phi(r_h) \cap (U_i - (U_i^* \cup \hat{U}_i)),$$

$$\tilde{r}_h = \begin{cases} r_h^+, & \text{if } x_i = u_i^+, \ \forall u_i \in \Phi(r_h), \\ r_h^-, & \text{otherwise}. \end{cases}$$

In the above, $U_i^\circ$ is the set of unassigned sub-causes of unassigned compound causes. The calculation of Eq. (39) includes procedures where states of compound causes in $U_c^\circ$ are tentatively given by assigning states to single causes in $U_i^\circ$, and the problem is re-simplified to the singly-connected. Thus the procedure needs as many re-simplifications as combinations of possible states of single causes in $U_i^\circ$.

6. Numerical example

Let us calculate the value of $\text{Pos}(\hat{P}, \hat{Q} | P^*, Q^*)$ in the problem of Fig. 3. The marginal possibility distributions of single causes and the conditional causal possibilities are given below:

- $\Pi(u_i^+) = 0.7$, $\Pi(u_i^-) = 1.0$, $\Pi(u_2^+) = 1.0$, $\Pi(u_2^-) = 0.4$, $\Pi(u_3^+) = 1.0$, $\Pi(u_3^-) = 1.0$:
- $\Pi(u_4^+) = 0.4$, $\Pi(u_4^-) = 1.0$, $\Pi(u_5^+) = 0.7$, $\Pi(u_5^-) = 1.0$:
- $\Pi(c_{1,2}^+ | u_i^+) = 0.7$, $\Pi(c_{1,2}^- | u_i^+) = 1.0$, $\Pi(c_{2,3}^+ | u_2^+) = 0.4$, $\Pi(c_{2,3}^- | u_2^+) = 1.0$:
- $\Pi(c_{3,2}^+ | u_3^+) = 0.4$, $\Pi(c_{3,2}^- | u_3^+) = 1.0$, $\Pi(c_{4,2}^+ | u_4^+) = 0.1$, $\Pi(c_{4,2}^- | u_4^+) = 1.0$:
- $\Pi(c_{4,4}^+ | u_4^+) = 1.0$, $\Pi(c_{4,4}^- | u_4^+) = 1.0$, $\Pi(c_{4,5}^+ | u_4^+) = 1.0$, $\Pi(c_{4,5}^- | u_4^+) = 0.7$:
- $\Pi(c_{5,4}^+ | u_5^+) = 0.7$, $\Pi(c_{5,4}^- | u_5^+) = 1.0$, $\Pi(c_{5,5}^+ | u_5^+) = 1.0$, $\Pi(c_{5,5}^- | u_5^+) = 0.7$. 

\( \Pi(c_{12,1}^+ | u_{12}^+) = 0.7, \quad \Pi(c_{12,1}^- | u_{12}^+) = 1.0, \quad \Pi(c_{34,3}^- | u_{34}^+) = 1.0, \quad \Pi(c_{34,3}^+ | u_{34}^+) = 0.4, \quad \Pi(c_{45,4}^+ | u_{45}^+) = 0.1, \quad \Pi(c_{45,4}^- | u_{45}^+) = 1.0. \)

In the CCC problem in Fig. 3, \( u_{45} \) is multiply-connected. Since the state of \( u_{45} \) only depends on that of \( u_{5} \), possible sets of tentatively assigned states of \( u_{5} \) and \( u_{45} \) are \( \hat{S}_1 = \{ u_{5}^+, u_{45}^+ \} \) and \( \hat{S}_2 = \{ u_{5}^-, u_{45}^- \} \). Then, we get the following from Eq. (39):

\[
\begin{align*}
\text{Pos}(\hat{P}, \hat{Q} | P^*, Q^*) &= \text{Pos}(\hat{S}_1 \cup \hat{P}, \hat{Q} | P^*, Q^*) \lor \text{Pos}(\hat{S}_2 \cup \hat{P}, \hat{Q} | P^*, Q^*), \\
\text{Pos}(\hat{S}_1 \cup \hat{P}, \hat{Q} | P^*, Q^*) &= \Pi(u_3^+ \land u_4^+ \land u_5^+ \land u_{34}^+ \land v_4^+ \land v_5^+ \land v_1^+ \land u_1^+ \land v_2^+ \land v_3^+), \\
\text{Pos}(\hat{S}_2 \cup \hat{P}, \hat{Q} | P^*, Q^*) &= \Pi(u_3^+ \land u_4^- \land u_5^- \land u_{34}^- \land v_4^- \land v_5^- \land v_1^- \land u_1^- \land v_2^- \land v_3^-). \\
\end{align*}
\]

(1) Calculation of Pos(\( \hat{S}_1 \cup \hat{P}, \hat{Q} | P^*, Q^* \))

Since we assume \( u_5 \) is positive, the problem in Fig. 3 is re-simplified to that in Fig. 4(a). Note that some conditional causal possibilities should be replaced as follows:

\( \Pi(c_{4,4}^+ | u_{4}^+) = 0.0, \quad \Pi(c_{4,4}^- | u_{4}^+) = 1.0, \quad \Pi(c_{5,4}^- | u_{5}^+) = 0.0, \quad \Pi(c_{5,4}^+ | u_{5}^+) = 1.0. \)

In this case,

\[
\begin{align*}
\Pi(v_2^+ | u_1^+ \land u_2^+ \land u_{12}^+) &= \Pi(v_2^+ | u_1^+ \land u_2^+ \land u_{12}^+) \land \Pi(v_3^+ | u_1^+ \land u_2^+ \land u_{12}^+) \land \Pi(v_4^- | u_1^+ \land u_2^+ \land u_{12}^+) \land \Pi(v_5^- | u_1^+ \land u_2^+ \land u_{12}^+). \\
\Pi(v_2^- | u_1^+ \land u_2^- \land u_{12}^+) &= \bigvee_{r_k \in \{ u_1, \ldots, u_5, u_{12}, u_{34}, u_{45} \}} \{ \Pi(c_{k,2}^+ | r_k^+) \land \Pi(r_k^+ | u_1^+ \land u_2^+ \land u_{12}^+) \} = 0.7. \\
\Pi(v_3^- | u_1^+ \land u_2^- \land u_{12}^+) &= \bigwedge_{r_k \in \{ u_1, \ldots, u_5, u_{12}, u_{34}, u_{45} \}} \{ (\Pi(c_{k,3}^- | r_k^-) \land \Pi(r_k^+ | u_1^+ \land u_2^+ \land u_{12}^+) ) \lor (\Pi(c_{k,5}^- | r_k^-) \land \Pi(r_k^- | u_1^+ \land u_2^- \land u_{12}^+) ) \} = 1.0. \\
\end{align*}
\]

Thus we get Pos(\( P^*, Q^* \)) = 0.7. Then,

\[
\begin{align*}
\text{Pos}(P^*, Q^*, \hat{S}_1 \cup \hat{P}, \hat{Q}) &= \Pi(v_1^+ \land v_2^+ \land v_3^+ \land u_3^+ \land u_5^+ \land u_1^+ \land u_2^+ \land u_{12}^+ \land u_{34}^+ \land u_{45}^+) \\
&\land \Pi(u_1^- \land u_2^+ \land u_3^+ \land u_4^+ \land u_5^+ \land u_{12}^- \land u_{34}^+ \land u_{45}^+) \\
&= \Pi(v_1^+ | u_1^+ \land u_2^+ \land u_3^+ \land u_4^+ \land u_5^+ \land u_{12} \land u_{34} \land u_{45}) \\
&\land \Pi(v_2^+ | u_1^+ \land u_2^+ \land u_3^+ \land u_4^+ \land u_5^+ \land u_{12} \land u_{34} \land u_{45}).
\end{align*}
\]
Thus from Eq. (38), we get $\text{Pos}(\hat{S}_1 \cup \hat{P}, \hat{Q} \mid P^*, Q^*) = 0.1$. 

\[
\wedge \Pi(v_3^+ \mid u_1^+ \land u_2^+ \land u_3^+ \land u_4^+ \land u_5^+ \land u_{12}^+ \land u_{34}^+ \land u_{45}^+)
\wedge \Pi(v_4^+ \mid u_1^+ \land u_2^+ \land u_3^+ \land u_4^+ \land u_5^+ \land u_{12}^+ \land u_{34}^+ \land u_{45}^+)
\wedge \Pi(v_5^+ \mid u_1^+ \land u_2^+ \land u_3^+ \land u_4^+ \land u_5^+ \land u_{12}^+ \land u_{34}^+ \land u_{45}^+)
\wedge \Pi(u_1^+) \land \pi(u_3^+) \land \pi(u_4^+) \land \pi(u_5^+)
= (1.0 \land 0.7 \land 1.0 \land 0.1 \land 0.7) \land (0.7 \land 1.0 \land 1.0 \land 0.4 \land 0.7)
= 0.1.
\]
Thus, we get

\[ \text{Pos}(\hat{S}_2 \cup \hat{P}, \hat{Q} | P^*, Q^*) \]

Since we assume \( u_5 \) is negative, the problem in Fig. (3) is re-simplified to that in Fig. 4(b). Thus, conditional causal possibilities between \( u_{45} \) and \( v_4 \) should be replaced by

\[
\Pi(c_{45}^+ | u_{45}^+) = 0.0, \quad \Pi(c_{454}^+ | u_{45}^+) = 1.0.
\]

In this case, we get

\[
\text{Pos}(P^*, Q^*) = \Pi(v_4^+ \land v_5^- | u_1^+ \land u_2^+ \land u_{12}^+) \land \Pi(u_1^+ \land u_2^+ \land u_{12}^+)
\]

\[
= \Pi(v_4^+ | u_1^+ \land u_2^+) \land \Pi(v_5^- | u_1^+ \land u_2^+ \land u_{12}^+) \land \Pi(u_1^+ \land u_2^+)
\]

\[
= 0.7,
\]

and

\[
\text{Pos}(R^*, P^*, \hat{S}_2 \cup \hat{P}, \hat{Q})
\]

\[
= \Pi(v_1^+ \land v_2^+ \land v_3^+ \land v_4^+ \land v_5^- | u_1^+ \land u_2^+ \land u_3^+ \land u_4^+ \land u_5^- \land u_{12}^+ \land u_{34}^+ \land u_{45}^-)
\]

\[
\land \Pi(u_1^+ \land u_2^+ \land u_3^+ \land u_4^+ \land u_5^- \land u_{12}^+ \land u_{34}^+ \land u_{45}^-)
\]

\[
= \Pi(v_1^+ | u_1^+ \land u_2^+ \land u_3^+ \land u_4^+ \land u_5^- \land u_{12}^+ \land u_{34}^+ \land u_{45}^-)
\]

\[
\land \Pi(v_2^+ | u_1^+ \land u_2^+ \land u_3^+ \land u_4^+ \land u_5^- \land u_{12}^+ \land u_{34}^+ \land u_{45}^-)
\]

\[
\land \Pi(v_3^+ | u_1^+ \land u_2^+ \land u_3^+ \land u_4^+ \land u_5^- \land u_{12}^+ \land u_{34}^+ \land u_{45}^-)
\]

\[
\land \Pi(v_4^+ | u_1^+ \land u_2^+ \land u_3^+ \land u_4^+ \land u_5^- \land u_{12}^+ \land u_{34}^+ \land u_{45}^-)
\]

\[
\land \Pi(v_5^- | u_1^+ \land u_2^+ \land u_3^+ \land u_4^+ \land u_5^- \land u_{12}^+ \land u_{34}^+ \land u_{45}^-)
\]

\[
\land \Pi(u_1^+ \land u_2^+) \land \Pi(u_3^+) \land \Pi(u_4^+) \land \Pi(u_5^-)
\]

\[
= (1.0 \land 0.7 \land 1.0 \land 0.7) \land (0.7 \land 1.0 \land 0.4 \land 1.0)
\]

\[
= 0.4.
\]

Thus, we get \( \text{Pos}(\hat{S}_2 \cup \hat{P}, \hat{Q} | P^*, Q^*) = 0.4 \) from Eq. (38), and \( \text{Pos}(\hat{P}, \hat{Q} | P^*, Q^*) = 0.4 \) is obtained as the result.

If the given value of \( \Pi(u_5^+) \) is changed from 0.4 to 0.7, \( \text{Pos}(R^*, P^*, \hat{S}_2 \cup \hat{P}, \hat{Q}) = 0.7 \) is obtained. In this case, Eq. (38) gives \( \text{Pos}(\hat{S}_2 \cup \hat{P}, \hat{Q} | P^*, Q^*) = 1.0 \) and we get \( \text{Pos}(\hat{P}, \hat{Q} | P^*, Q^*) = 1.0 \) as the solution.

7. Concluding remarks

At the end of the paper, let us mention briefly the CCC problem with higher-order compound causes, which are compound causes including other compound ones as their constituents. Though the paper does not cover such higher-order causal models, it seems that the proposed approach can be expanded to cope with them.

Look at Fig. 5 for example. In the figure, \( u_i (i = 1, \ldots, 5) \) are single causes, \( u_{12}, u_{34} \) are minimal compound causes, and \( u_{123}, u_{1234}, u_{1235} \) are the higher-order compound causes. \( v_j \) is an effect. Then, suppose that states of \( u_1, u_2 \) and \( u_{12} \) are observed to be positive. The hypothesis is that \( u_3, u_{123} \) and \( v_j \) are positive and \( u_4, u_{34} \) and \( u_{1234} \) are negative. We have no interest in \( u_5 \) and \( u_{1235} \) here.
Then, the simplification discussed in Section 5.1 is applicable to the example, if Assumption 4.2 is expanded so that single and compound causes composing a positive higher-order compound cause could not generate the effect that has a causal link with the higher-order compound cause. In this case the causal links from $u_1$ and $u_2$ to $v_j$ can be removed, because $u_{123}$ is positive and has a causal link with $v_j$. $u_{34}$, $u_{1234}$ and causal links from the nodes can also be removed, because those compound causes are negative. As the result, node $u_{123}$ becomes singly-connected after the simplification.

The problem itself is still multiply-connected, however it can be converted to the singly-connected by tentatively assigning possible states to $u_5$ and $u_{1235}$, and by re-simplifying it. Thus, the possibility of the hypothesis conditioned by the observed states can be calculated in the same way as shown in Section 5.3.

The paper developed the compound causal model to deal with the compound effects by multiple causes on uncertainty of causalities. Then, it applied the causal model to the CCC problem, and proposed a way to solve the problem.

As mentioned in the Introduction, conditional probabilities and possibilities have been used to express uncertainty of causalities. However, since these values do not express the degrees of uncertainty of causalities, conditional causal probabilities were proposed by [17,18], and conditional causal possibilities by [19–21]. The proposed expressions also have an advantage in the number of
probabilistic/possibilistic values that should be given as a priori knowledge. The weak point that compound causalities could not be dealt with was the compensation for the advantage.

The paper proposed a way to get rid of the compensation. The conventional conditional possibilities may be thought to be easier, when the compound causalities should be taken into account. However, in many applications that need this kind of approach, the number of combinations producing the compound effects is far smaller than that of all combinations of possible causes. In the case, the information that should be given as a priori knowledge is much smaller than that of conventional approaches with conditional possibilities.

Since the necessary information is marginal possibilities of single causes \((\Pi(u^+_i), \Pi(u^-_i))\) and conditional causal possibilities \((\Pi(c^+_{i,j} | r^+_k), \Pi(c^-_{i,j} | r^-_k))\), the total number of possibilistic values that should be given is \(2(|U| + |V| \cdot |R|) = 2(I + J \cdot K)\). \(K\) is the sum of the numbers of single causes and compound causes, and satisfies \(I \leq K \leq 2^I - 1\). \(K = 2^I - 1\) is the worst case where all combinations of single causes are compound causes including higher-order compound ones, and is as many as the number of conditional possibilities \(\Pi(v^+_j | u^+_i \land \ldots \land u^+_I), v^+_j \in \{v^+_j, v^-_j\}, u^+_i \in \{u^+_i, u^-_i\}\). However, if the number of combinations producing the compound effects is far smaller than \(2^I - 1\), the drastic reduction of the necessary information is possible and the proposed approach becomes more practical than those with conventional conditional possibilities.

Appendix A. Proof of Eqs. (17a) and (17b)

Eq. (17a) is proved as follows, using the definition of joint-causation event and the possibilistic causation independence of causation events:

\[
\Pi(c^+_{g,h,j} | u^+_{g,h}) = \Pi \left( \bigvee_{i=g,h} c^+_{i,j} \land u^+_{g,h} \bigg| u^+_{g,h} \right) = \Pi \left( \bigvee_{i=g,h} c^+_{i,j} \bigg| u^+_{g,h} \land u^+_{g,h} \right) \land \Pi(u^+_{g,h} | u^+_{g,h})
\]

\[
= \Pi \left( \bigvee_{i=g,h} c^+_{i,j} \bigg| u^+_{g,h} \right) = \bigvee_{i=g,h} \Pi(c^+_{i,j} | u^+_{g,h}) = \bigvee_{i=g,h} \Pi(c^+_{i,j} | u^+_i). \quad \square
\]

For Eq. (17b), the next logical formula is used:

\[
c^-_{g,h,j} \land u^+_{g,h} \leftrightarrow \left\{ \bigvee_{i=g,h} c^+_{i,j} \land u^+_{g,h} \right\} \land u^+_{g,h} \leftrightarrow \left\{ \bigvee_{i=g,h} c^+_{i,j} \lor u^-_{g,h} \right\} \land u^+_{g,h}
\]

\[
\leftrightarrow \left\{ \bigvee_{i=g,h} c^+_{i,j} \land u^+_{g,h} \right\} \lor (u^-_{g,h} \land u^+_{g,h}) \leftrightarrow \bigwedge_{i=g,h} c^-_{i,j} \land u^+_{g,h}.
\]
This formula gives \( \Pi(c_{g,h,j}^- \land u_{g,h}^+) = \Pi( \bigwedge_{i=g,h} c_{i,j}^- \land u_{g,h}^+) \). Thus, the next equation is derived using Eq. (11):

\[
\Pi(c_{g,h,j}^- | u_{g,h}^+) = \Pi \left( \bigwedge_{i=g,h} c_{i,j}^- \bigg| u_{g,h}^+ \right).
\]

Then, it is proved that the next equation holds (see [22]):

\[
\Pi \left( \bigwedge_{i=g,h} c_{i,j}^- \bigg| u_{g,h}^+ \right) = \bigwedge_{i=g,h} \Pi(c_{i,j}^- | u_{g,h}^+) .
\]

From the above and the possibilistic causation independence of causation events, Eq. (17b) is proved as follows:

\[
\Pi(c_{g,h,j}^- | u_{g,h}^+) = \bigwedge_{i=g,h} \Pi(c_{i,j}^- | u_{g,h}^+) = \bigwedge_{i=g,h} \Pi(c_{i,j}^- | u_{i}^+). \hspace{1cm} \square
\]

**Appendix B. Proof of formula (21)**

Since \( \overline{\alpha(u_i,v_j)} \rightarrow c_{i,j}^- \) holds, it is easily proved that the next formula holds:

\[
\bigvee_{u_i \in U_i} c_{i,j}^+ \leftrightarrow c_{g,h,j}^+ \bigwedge \bigvee_{u_i \in \{u_i | \overline{\alpha(u_i,v_j)} \cap U_i \}} \\
\bigvee_{u_i \in \{u_i | \overline{\alpha(u_i,v_j)} \cap U_i \}} c_{i,j}^+ \leftrightarrow \bigvee_{u_i \in \{u_i | \overline{\alpha(u_i,v_j)} \cap U_i \}} c_{i,j}^+.
\]

Then, the next formula holds from formula (16):

\[
\bigvee_{u_{g,h,j} \in U_i} c_{g,h,j}^+ \leftrightarrow \bigvee_{u_{g,h} \in \{u_{g,h} | \beta(u_{g,h},v_j) \cap U_{g,h} \}} c_{g,h,j}^+ \\
\bigvee_{u_{g,h} \in \{u_{g,h} | \beta(u_{g,h},v_j) \cap U_{g,h} \}} \left\{ \bigvee_{u_i \in \Phi(u_{g,h})} c_{i,j}^+ \bigwedge u_{g,h}^+ \right\}.
\]

Thus, formula (21) is derived from formula (20) as follows:

\[
v_j^+ \leftrightarrow \bigvee_{u_i \in U_i} c_{i,j}^+ \bigwedge c_{g,h,j}^+ \\
\bigvee_{u_{g,h} \in \{u_{g,h} | \beta(u_{g,h},v_j) \cap U_{g,h} \}} \left\{ \bigvee_{u_i \in \Phi(u_{g,h})} c_{i,j}^+ \bigwedge u_{g,h}^+ \right\} \\
\leftrightarrow \bigvee_{u_{g,h} \in \{u_{g,h} | \beta(u_{g,h},v_j) \cap U_{g,h} \}} c_{g,h,j}^+ \bigwedge u_{g,h}^+.
\]
The case where \( c_{k_1,j_1} \land c_{k_2,j_2} \) is contradictory is only the case where \( k_1 = k_2 \), \( j_1 = j_2 \) and \( c_{k_1,j_1} \leftrightarrow \overline{c_{k_2,j_2}} \). Thus, if \( k_1 = k_2 \) and \( j_1 = j_2 \), \( c_{k_1,j_1} \leftrightarrow c_{k_2,j_2} \) must hold and the lemma holds evidently. If \( \Pi(\omega) = 0 \), the lemma also holds. In the other cases, the following holds from Eqs. (9) and (12):

\[
\Pi(c_{k_1,j_1} \land c_{k_2,j_2} | \omega) = \bigvee_{w_{k_1} \in \{r_{k_1,j_1}^+, r_{k_1,j_1}^-\}} \{ \Pi(c_{k_1,j_1}^* c_{k_2,j_2}^* w_{k_1} | \omega) \} = \bigvee_{w_{k_1} \in \{r_{k_1,j_1}^+, r_{k_1,j_1}^-\}} \{ \Pi(c_{k_1,j_1}^* c_{k_2,j_2}^* w_{k_1} | \omega) \land \Pi(c_{k_2,j_2}^* w_{k_1} | \omega) \land \Pi(w_{k_1} | \omega) \}.
\]

If \( c_{k_2,j_2} \land r_{k_1}^+ \land \omega \) is not contradictory, \( c_{k_2,j_2}^* \land \omega \) is a context of \( c_{k_1,j_1} \) and \( \Pi(c_{k_1,j_1}^* c_{k_2,j_2}^* r_{k_1}^+ \land \omega) = \Pi(c_{k_1,j_1}^* r_{k_1}^+ \land \omega) \) holds, because the CCC problem is singly-connected. It is easily proved that \( \Pi(c_{k_1,j_1}^* c_{k_2,j_2}^* r_{k_1}^- \land \omega) = \Pi(c_{k_1,j_1}^* r_{k_1}^- \land \omega) \) also holds, if \( c_{k_2,j_2} \land r_{k_1}^- \land \omega \) is not contradictory. In the case where \( c_{k_2,j_2} \land r_{k_1}^- \land \omega \) is contradictory \( (w_{k_1} \in \{r_{k_1,j_1}^+, r_{k_1,j_1}^-\}) \), \( \Pi(c_{k_2,j_2}^* w_{k_1} \land \omega) \land \Pi(w_{k_1} | \omega) = 0 \) holds, because \( \Pi(\omega) \neq 0 \) is assumed here. Thus, the following is derived.

\[
\bigvee_{w_{k_1}} \{ \Pi(c_{k_1,j_1}^* c_{k_2,j_2}^* w_{k_1} | \omega) \land \Pi(c_{k_2,j_2}^* w_{k_1} | \omega) \land \Pi(w_{k_1} | \omega) \} = \bigvee_{w_{k_1}} \{ \Pi(c_{k_1,j_1}^* w_{k_1} | \omega) \land \Pi(c_{k_2,j_2}^* w_{k_1} | \omega) \land \Pi(w_{k_1} | \omega) \} = \bigvee_{w_{k_1}} \{ \Pi(c_{k_2,j_2}^* w_{k_1} | \omega) \land \Pi(c_{k_1,j_1}^* w_{k_1} | \omega) \} = \Pi(c_{k_1,j_1}^* | \omega) \land \Pi(c_{k_2,j_2}^* | \omega). \quad \square
\]

Appendix C. Proof of Lemma 5.1
Appendix D. Proof of Lemma 5.2

The next equation is derived from Eqs. (9) and (12):

\[
\Pi(c_{k,j}^*|\omega) = \bigvee_{w_k \in \{r_k^+ \cup r_k^-\}} \Pi(c_{k,j}^* \land w_k|\omega) = \bigvee_{w_k \in \{r_k^+ \cup r_k^-\}} \{\Pi(c_{k,j}^*|w_k \land \omega) \land \Pi(w_k|\omega)\}.
\]

If \(r_k^+ \land \omega\) is not contradictory, \(\omega\) is a context of \(c_{k,j}\) and \(\Pi(c_{k,j}^*|r_k^+ \land \omega) = \Pi(c_{k,j}^*|r_k^+)\) holds, because the CCC problem is singly-connected. It is easily proved that \(\Pi(c_{k,j}^*|r_k^- \land \omega) = \Pi(c_{k,j}^*|r_k^-)\) holds, if \(r_k^- \land \omega\) is not contradictory. In the case where \(w_k \land \omega\) is contradictory, \(\Pi(w_k|\omega) = 0\) holds because \(\Pi(\omega) \neq 0\) is assumed. Thus, the lemma is proved as follows:

\[
\Pi(c_{k,j}^*|\omega) = \bigvee_{w_k \in \{r_k^+ \cup r_k^-\}} \{\Pi(c_{k,j}^*|w_k \land \omega) \land \Pi(w_k|\omega)\}. \quad \square
\]

Appendix E. Proof of Proposition 5.1

Let \(U(v_j) = \{r_k | z(r_k, v_j) \lor \beta(r_k, v_j), r_k \in R\}\). Then

\[
\Pi(v_j^+|\omega) = \Pi \left( \bigvee_{r_k \in U(v_j)} c_{k,j}^+ \bigg| \omega \right) = \bigvee_{r_k \in U(v_j)} \Pi(c_{k,j}^+|\omega)
\]

is derived from formula (21). Thus, by applying Lemma 5.2, the next equation is obtained.

\[
\Pi(v_j^+|\omega) = \bigvee_{w_k \in \{r_k^+ \cup r_k^-\}} \Pi(c_{k,j}^+|w_k \land \omega) \land \Pi(w_k|\omega)\}.
\]

It is easily proved that \(\Pi(c_{k,j}^+|r_k^-) \land \Pi(r_k^-|\omega) = 0\) holds for any \(r_k^-, 1 \leq k \leq K\), because \(\Pi(\omega) \neq 0\). Thus, Eq. \((29a)\) is derived from the above equation.

Then, the next equation is derived from Eq. (21) and Lemma 5.1:

\[
\Pi(v_j^-|\omega) = \Pi \left( \bigwedge_{r_k \in U(v_j)} c_{k,j}^- \bigg| \omega \right) = \bigwedge_{r_k \in U(v_j)} \Pi(c_{k,j}^-|\omega)
\]

Eq. \((29b)\) is obtained by applying Lemma 5.2 to the above equation. \(\square\)

Appendix F. Proof of Proposition 5.2

The case where \(v_{j_1}^* \land v_{j_2}^*\) is contradictory is only the case where \(j_1 = j_2\) and \(v_{j_1}^* \leftrightarrow v_{j_2}^*\). Thus, if \(j_1 = j_2\), \(v_{j_1}^* \leftrightarrow v_{j_2}^*\) must hold and the proposition holds evidently. If \(\Pi(\omega) = 0\), the proposition also
holds. In the other cases, the proposition is proved as follows:

1. When \( v_j^* \leftrightarrow v_{j_1}^+ \), \( v_{j_2}^* \leftrightarrow v_{j_2}^- \), the following equation is derived using formula (20):

\[
\Pi(v_{j_1}^+ \land v_{j_2}^- | \omega) = \Pi \left( \bigvee_k \left( c_{k,j_1}^+ \land \bigwedge_h c_{h,j_2}^- \right) | \omega \right) = \Pi \left( \bigvee_{k,h} (c_{k,j_1}^+ \land c_{h,j_2}^+) | \omega \right)
\]

Using Lemma 5.1 and Formula (20), the proposition is proved as follows:

\[
\bigvee_{k,h} \Pi(c_{k,j_1}^+ \land c_{h,j_2}^+ | \omega) = \bigvee_{k,h} \{ \Pi(c_{k,j_1}^+ | \omega) \land \Pi(c_{h,j_2}^+ | \omega) \}
\]

\[
= \bigvee_k \Pi(c_{k,j_1}^+ | \omega) \land \bigvee_h \Pi(c_{h,j_2}^+ | \omega)
\]

\[
= \Pi \left( \bigvee_k c_{k,j_1}^+ | \omega \right) \land \Pi \left( \bigvee_h c_{h,j_2}^+ | \omega \right) = \Pi(v_{j_1}^+ | \omega) \land \Pi(v_{j_2}^+ | \omega).
\]

2. When \( v_j^* \leftrightarrow v_{j_1}^+ \), \( v_{j_2}^* \leftrightarrow v_{j_2}^- \), formula (20) gives the following equation:

\[
\Pi(v_{j_1}^+ \land v_{j_2}^- | \omega) = \Pi \left( \bigvee_k \left( c_{k,j_1}^+ \land \bigwedge_h c_{h,j_2}^- \right) | \omega \right)
\]

\[
= \bigvee_k \Pi(c_{k,j_1}^+ \land \bigwedge_h c_{h,j_2}^- | \omega)
\]

Then, Lemma 5.1 and formula (20) derive the next equation:

\[
\bigvee_k \Pi \left( c_{k,j_1}^+ \land \bigwedge_h c_{h,j_2}^- \right) | \omega) = \bigvee_k \left\{ \Pi(c_{k,j_1}^+ | \omega) \land \bigwedge_h \Pi(c_{h,j_2}^- | \omega) \right\}
\]

\[
= \bigvee_k \Pi(c_{k,j_1}^+ | \omega) \land \bigwedge_h \Pi(c_{h,j_2}^- | \omega)
\]

\[
= \Pi \left( \bigvee_k c_{k,j_1}^+ | \omega \right) \land \Pi \left( \bigwedge_h c_{h,j_2}^- | \omega \right)
\]

\[
= \Pi(v_{j_1}^+ | \omega) \land \Pi(v_{j_2}^- | \omega).
\]

3. When \( v_j^* \leftrightarrow v_{j_1}^- \), \( v_{j_2}^* \leftrightarrow v_{j_2}^- \), the next equation is proved by formula (20) and Lemma 5.1:

\[
\Pi(v_{j_1}^- \land v_{j_2}^- | \omega) = \Pi \left( \bigwedge_k c_{k,j_1}^- \land \bigwedge_h c_{h,j_2}^- \right) | \omega)
\]

\[
= \bigwedge_k \Pi(c_{k,j_1}^- | \omega) \land \bigwedge_h \Pi(c_{h,j_2}^- | \omega)
\]

\[
= \Pi \left( \bigwedge_k c_{k,j_1}^- | \omega \right) \land \Pi \left( \bigwedge_h c_{h,j_2}^- | \omega \right) = \Pi(v_{j_1}^- | \omega) \land \Pi(v_{j_2}^- | \omega).
\]
Appendix G. Proof of Proposition 5.3

If \( \Pi(\omega) = 0 \), the proposition holds evidently. It also holds obviously in the case where \( r_k \) is a single cause. In the other cases, let \( r_k^+ \leftrightarrow u_i^+ \land u_{g,h}^- \). Then, the next equation is derived from Eq. (12):

\[
\Pi(r_k^+ | \omega) = \Pi(u_i^+ \land u_{g,h}^- | \omega) = \Pi(u_i^+ | u_{g,h}^- \land \omega) \land \Pi(u_{g,h}^- | \omega).
\]

(1) In the case where \( \omega \to u_i^+ \), \( \Pi(\omega \to u_i^+) = \Pi(\omega \land u_i^+) = \Pi(u_i^+ | \omega) \land \Pi(\omega) = 0 \). Since \( \Pi(\omega) \neq 0 \) here, \( \Pi(u_i^- | \omega) = 0 \) and \( \Pi(u_i^+ | \omega) = 1 \) are derived. On the other hand, \( \{u_i^+ \land \omega\} \to u_i^+ \) holds, because \( \omega \to u_i^+ \). Thus, \( \Pi(u_i^+ | \omega) = \Pi(u_i^+ | u_{g,h}^- \land \omega) = 1 \) and \( \Pi(u_i^+ \land u_{g,h}^- | \omega) = \Pi(u_i^+ | \omega) \land \Pi(u_{g,h}^- | \omega) \) hold.

(2) In the case where \( \omega \to u_i^- \), \( \Pi(\omega \to u_i^-) = \Pi(\omega \land u_i^-) = \Pi(u_i^- | \omega) \land \Pi(\omega) = 0 \). Since \( \Pi(\omega) \neq 0 \), \( \Pi(u_i^- | \omega) = 0 \) is derived. On the other hand, \( \{u_i^+ \land \omega\} \to u_i^- \) holds, because \( \omega \to u_i^- \). Thus, if \( u_i^+ \land \omega \) is not contradictory, \( \Pi(u_i^+ | \omega) = \Pi(u_i^+ | u_{g,h}^- \land \omega) = 0 \) holds. If \( u_i^+ \land \omega \) is contradictory, \( \Pi(u_i^+ \land u_{g,h}^- | \omega) = \Pi(u_{g,h}^- | \omega) = 0 \). Therefore, \( \Pi(u_i^+ \land u_{g,h}^- | \omega) = \Pi(u_i^- | \omega) \land \Pi(u_{g,h}^- | \omega) \) holds.

(3) In the other case, \( u_i^+ \) and \( \omega \) are possibilistically independent of each other, because \( \omega \) is a conjunction of states of causes other than \( u_i \). It is also evident that \( u_i^+ \) and \( u_{g,h}^- \) are possibilistically independent of each other. Thus, \( \Pi(u_i^+ | \omega) = \Pi(u_i^+ | u_{g,h}^- \land \omega) = \Pi(u_i^+ | \omega) \land \Pi(u_{g,h}^- | \omega) \) hold.

From the above, it is proved that \( \Pi(u_i^+ \land u_{g,h}^- | \omega) = \Pi(u_i^+ | \omega) \land \Pi(u_{g,h}^- | \omega) \) in general. By applying this equation recursively, Eq. (31a) is proved. Then, Eq. (31b) is proved as follows:

\[
\Pi(r_k^- | \omega) = \Pi \left( \bigvee_{u_i \in \Phi(r_k^\perp)} u_i \biggm| \omega \right) = \bigvee_{u_i \in \Phi(r_k^\perp)} \Pi(u_i | \omega).
\]

Appendix H. Proof of Lemma 5.3

Let \( D = \Phi(r_k^\perp) \cap \Phi(r_k^\perp) \), \( A = \Phi(r_k^\perp) - D = \Phi(r_k^\perp) - \Phi(r_k^\perp) \) and \( B = \Phi(r_k^\perp) - D = \Phi(r_k^\perp) - \Phi(r_k^\perp) \). Also let \( r_a^+ \leftrightarrow \bigwedge_{u_i \in A} u_i^+ \), \( r_b^+ \leftrightarrow \bigwedge_{u_i \in B} u_i^+ \) and \( r_a^- \leftrightarrow \bigwedge_{u_i \in D} u_i^- \). If \( A, B \) or \( D \) is empty, \( r_a^+, r_b^+ \) or \( r_a^- \) is a tautology, respectively. Then, \( r_k^\perp \leftrightarrow r_a^+ \land r_a^- \) and \( r_k^\perp \leftrightarrow r_a^+ \land r_a^- \) hold.

(1) It is obvious that Eq. (33a) holds due to Eq. (23a).

(2) In the case of Eq. (33b),

\[
\Pi(r_k^\perp \land r_k^\perp) = \Pi(r_a^+ \land r_a^- \land r_b^+ \land r_b^-) = \Pi((r_a^- \lor r_a^+) \land (r_b^- \lor r_b^+)) = \Pi(r_a^- \land r_b^-) \lor \Pi(r_a^+ \land r_b^+).
\]

It is easily proved that \( r_a^- \) and \( r_b^- \) are non-interactive, because they have no single cause in common. Thus,

\[
\Pi(r_a^- \lor r_b^-) = \Pi(r_a^-) \lor \Pi(r_b^-) = (\Pi(r_a^-) \lor \Pi(r_a^+)) \land (\Pi(r_b^-) \lor \Pi(r_b^+)) = \Pi(r_a^+ \land r_a^-) \land \Pi(r_b^+ \land r_b^-) = \Pi(r_a^+ \land r_a^-) \land \Pi(r_b^+ \land r_b^-) = \Pi(r_k^\perp) \land \Pi(r_k^\perp).
\]
(3) In the case of Eq. (33c),

\[ \Pi(r_{k_1}^+ \land r_{k_2}^-) = \Pi(r_{d_a}^+ \land r_{k_1}^+ \land r_{d_b}^-) = \Pi(r_{d_a}^+ \land r_{d_b}^- \lor r_{d_d}^-) = \Pi(r_{d_a}^+ \land r_{d_b}^- \land r_{d_d}^-) = \Pi(r_{d_a}^+ \land r_{d_b}^- \land r_{b_b}^-). \]

\( r_{d_a}^+ \land r_{d_b}^- \) and \( r_{b_b}^- \) are non-interactive, because they have no single cause in common. Thus,

\[ \Pi(r_{d_a}^+ \land r_{d_b}^- \land r_{b_b}^-) = \Pi(r_{d_a}^+ \land r_{d_b}^-) \land \Pi(r_{b_b}^-) = \Pi(r_{k_1}^+) \land \Pi(r_{k_2}^-) \]

\[ = \Pi(r_{k_1}^+) \land \Pi \left( \bigwedge_{u_i \in \Phi (r_{k_2}^-) \setminus \Phi (r_{k_1}^+)} u_i^+ \right) = \Pi(r_{k_1}^+) \land \Pi \left( \bigvee_{u_i \in \Phi (r_{k_2}^+) \setminus \Phi (r_{k_1}^-)} u_i^- \right) \]

\[ = \Pi(r_{k_1}^+) \land \bigvee_{u_i \in \Phi (r_{k_2}^+) \setminus \Phi (r_{k_1}^-)} \Pi(u_i^-). \]

\[ \square \]

Appendix I. Proof of Proposition 5.4

Let \( u^p \leftrightarrow \bigwedge_{u_i \in \Phi (R_p)} u_i^+ \leftrightarrow \bigwedge_{r_h \in R^p} r_h^- \). Then, the next logical formula holds:

\[ \bigwedge_{r_h \in R^p} r_h^+ \land \bigwedge_{r_h \in R^p} r_h^- \leftrightarrow u^p \land \bigwedge_{r_h \in R^p} r_h^- \leftrightarrow u^p \land \bigwedge_{r_h \in R^p} r_h^- \setminus R_p. \]

Thus,

\[ \Pi \left( \bigwedge_{r_h \in R^p} r_h^+ \land \bigwedge_{r_h \in R^p} r_h^- \right) = \Pi \left( u^p \land \bigwedge_{r_h \in R^p} r_h^- \setminus R_p \right). \]

\( u^p \) and \( \bigwedge_{r_h \in R^p} r_h^- \setminus R_p \) are non-interactive, because they have no common single cause. Thus,

\[ \Pi \left( u^p \land \bigwedge_{r_h \in R^p} r_h^- \setminus R_p \right) = \Pi \left( \bigwedge_{r_h \in R^p} r_h^- \setminus R_p \right) \land \Pi(u^p). \]

From Lemma 5.3, Eq. (34) is derived:

\[ \Pi \left( \bigwedge_{r_h \in R^p} r_h^- \setminus R_p \right) \land \Pi(u^p) = \Pi \left( \bigwedge_{r_h \in R^p} r_h^- \setminus R_p \right) \land \Pi \left( \bigwedge_{r_h \in R^p} r_h^+ \right) = \bigwedge_{r_h \in R^p} \Pi(r_h^- \setminus R_p) \land \bigwedge_{r_k \in R^p} \Pi(r_k^+). \]

\[ \square \]

References